On Certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion due to the Direct Action of the Planets.

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Hansen has characterized the calculation of the coefficients of these inequalities as extremely difficult. However, it seems to me that, if the shortest methods are followed, there is no ground for such an assertion. The work may be divided into two portions, independent of each other. In one the object is to develop, in periodic series, certain functions of the moon's coordinates, which in number do not exceed five. This portion is the same whatever planet may be considered to act, and hence may be done once for all. In the other portion we seek the coefficients of certain terms in the periodic development of certain functions, five also in number, which involve the coordinates of the earth and planet only. And this part of the work is very similar to that in which the perturbations of the earth by the planet in question are the things sought. And as the multiples of the mean motions of these two bodies, which enter into the expression of the argument of the inequalities under consideration, are necessarily quite large, approximative values of the coefficients may be obtained by semi-convergent series similar to the well-known theorem of Stirling. This matter was first elaborated by Cauchy,* but, in the method as left by him, we are directed to compute special values of the successive derivatives of the functions to be developed. Now it unfortunately happens that these functions are enormously complicated by successive differentiation, so that it is almost impossible to write at length their second derivatives. Manifestly then it would

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* *Mémoire sur les approximations des fonctions de très-grands nombres;* and *Rapport sur un Mémoire de M. Le Verrier, qui a pour objet la détermination d'une grande inégalité du moyen mouvement de la planète Pallas*: *Comptes Rendus de l'Académie des Sciences de Paris*, *Tom.* XX, *pp.* 691–726, 767–786, 825–847.

be a great saving of labor to substitute for the computation of special values of these derivatives a computation of a certain number of special values of the original function, distributed in such a way that the maximum advantage may be obtained. This modification has given rise to an elegant piece of analysis. It will be noticed that, in this method, it is necessary to substitute in the formulæ, from the outset, the numerical values of the elements of the orbits of the earth and planet. There seems to be no objection to this on the practical side, as, for the computation of the inequalities sought, no partial derivatives of R, with respect to these elements, are required.

I.

If the masses of the moon, earth and the planet considered are denoted severally by m, M and m'', and the geocentric rectangular coordinates of the moon by x, y, and z, the similar coordinates of the sun by x', y' and z', and the heliocentric coordinates of the planet by x'', y'' and z'', the perturbative function, for the direct action of the planet on the moon, is

$$R = m''\left[\frac{1}{[(x''+x'-x)^2+(y''+y'-y)^2+(z''+z'-z)^2]^{\frac{1}{2}}} - \frac{(x''+x')x+(y''+y')y+(z''+z')z}{[(x''+x')^2+(y''+y')^2+(z''+z')^2]^{\frac{3}{2}}}\right].$$

But, by a slight substitution in and modification of this expression, we take account of the lunar perturbations of the solar coordinates. Let X, Y and Z denote the coordinates of the sun referred to the centre of gravity of the earth and moon, we shall then have

$$x' = X + \frac{m}{M+m}x, \quad y' = Y + \frac{m}{M+m}y, \quad z' = Z + \frac{m}{M+m}z.$$

And Δ may denote the distance of the planet from the centre of gravity of the earth and moon, so that

$$\Delta^2 = (x''+X)^2 + (y''+Y)^2 + (z''+Z)^2,$$

also r the radius vector of the moon, so that

$$r^2 = x^2 + y^2 + z^2;$$

moreover, for brevity, put

$$P = (x''+X)x + (y''+Y)y + (z''+Z)z.$$

Then R takes the form

$$R = m''\left[\frac{1}{\left[\Delta^2 - 2\dfrac{M}{M+m}P + \dfrac{M^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}} - \frac{P + \dfrac{m}{M+m}r^2}{\left[\Delta^2 + 2\dfrac{m}{M+m}P + \dfrac{m^2}{(M+m)^2}r^2\right]^{\frac{3}{2}}}\right].$$

But it is evident that this expression, differentiated with respect to the variables x, y and z, will not furnish differential coefficients identical in value with those the expression gives before the transformation, as x', y' and z' have now been made to involve x, y and z. But a little consideration shows the modification which will remedy this. It is plain we ought to multiply the first term by $\frac{M+m}{M}$, and, multiplying the last term by $-\frac{M+m}{m}$, substitute unity for the numerator and reduce the exponent of the denominator from $\frac{3}{2}$ to $\frac{1}{2}$.

Thus the proper form of R is

$$R = m''\left[\frac{M+m}{M}\frac{1}{\left[\Delta^2 - 2\frac{M}{M+m}P + \frac{M^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}} + \frac{M+m}{m}\frac{1}{\left[\Delta^2 + 2\frac{m}{M+m}P + \frac{m^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}}\right].$$

When this expression is expanded in a series proceeding according to ascending powers of the lunar coordinates, and the terms independent of the latter omitted, we get

$$\begin{aligned} R = m''\Bigg\{ & \frac{4.3}{2.4}\frac{P^2}{\Delta^5} - \frac{2}{1}\cdot\frac{2.1}{2.4}\frac{r^2}{\Delta^3} \\ & + \frac{M^2-m^2}{(M+m)^2}\left[\frac{6.5.4}{2.4.6}\frac{P^3}{\Delta^7} - \frac{3}{1}\cdot\frac{4.3.2}{2.4.6}\frac{Pr^2}{\Delta^5}\right] \\ & + \frac{M^3+m^3}{(M+m)^3}\left[\frac{8.7.6.5}{2.4.6.8}\frac{P^4}{\Delta^9} - \frac{4}{1}\cdot\frac{6.5.4.3}{2.4.6.8}\frac{P^2r^2}{\Delta^7} + \frac{4.3}{1.2}\cdot\frac{4.3.2.1}{2.4.6.8}\frac{r^4}{\Delta^5}\right] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \Bigg\}. \end{aligned}$$

The terms of this series follow a quite evident law, and it is easy to write as many as there may be occasion for. But, hitherto, no sensible inequalities have been found arising from the terms beyond the first line. This line furnishes all the inequalities which are not factored by the small ratio $\frac{a}{a'}$, whose value is about $\frac{1}{400}$. And the following two lines of terms can add to the coefficients of these only parts which have the very small factor $\frac{a^2}{a'^2}$. For these reasons we can restrict ourselves to the first line of terms, and write very simply

$$R = m''\left[\frac{3}{2}\frac{P^2}{\Delta^5} - \frac{1}{2}\frac{r^2}{\Delta^3}\right].$$

Restoring the equivalent of P,

$$R = m''\Big\{\Big[\frac{3}{2}\frac{(x''+X)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]x^2+\Big[\frac{3}{2}\frac{(y''+Y)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]y^2$$
$$+\Big[\frac{3}{2}\frac{(z''+Z)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]z^2+3\frac{(x''+X)(y''+Y)}{\varDelta^5}xy$$
$$+3\frac{(x''+X)(z''+Z)}{\varDelta^5}xz+3\frac{(y''+Y)(z''+Z)}{\varDelta^5}yz\Big\}.$$

This expression has the advantage of exhibiting the value of R as a sum of terms of which each is the product of two factors, one of which depends solely on the coordinates of the moon and the other is independent of them.

If we denote the factors of x^2, y^2 and z^2 in R severally by A, B and C, we shall have the relation $\quad A+B+C=0$.

Hence it is plain that the number of terms can be reduced from six to five. As we shall take the ecliptic for the plane of xy, we will have $Z=0$. We can then write

$$R = m''\Big\{\frac{1}{4}\Big[\frac{1}{\varDelta^3}-3\frac{z''^2}{\varDelta^5}\Big](r^2-3z^2)+\frac{3}{4}\frac{(y''+Y)^2-(x''+X)^2}{\varDelta^5}(y^2-x^2)$$
$$+3\frac{(x''+X)(y''+Y)}{\varDelta^5}xy+3\frac{(x''+X)z''}{\varDelta^5}xz+3\frac{(y''+Y)z''}{\varDelta^5}yz\Big\}.$$

II.

We will now express the five factors of the terms of R, viz. r^2-3z^2, x^2-y^2, xy, xz, and yz, as functions of t, the time, when elliptic values are attributed to the coordinates, leaving, however, the longitudes of the perigee and node indeterminate, so that the latter may have their motions proportional to t.

Using Delaunay's notation, and, in addition, putting v for the true anomaly, we have

$$x = r\cos(v+g)\cos h-(1-2\gamma^2)r\sin(v+g)\sin h,$$
$$y = r\cos(v+g)\sin h+(1-2\gamma^2)r\sin(v+g)\cos h,$$
$$z = 2\gamma\sqrt{1-\gamma^2}\,r\sin(v+g);$$

or, in a slightly different form,

$$x = (1-\gamma^2)r\cos(v+g+h)+\gamma^2 r\cos(v+g-h),$$
$$y = (1-\gamma^2)r\sin(v+g+h)-\gamma^2 r\sin(v+g-h),$$
$$z = 2\gamma\sqrt{1-\gamma^2}\,r\sin(v+g).$$

From these equations we derive

$$z^2 = 2\gamma^2(1-\gamma^2)r^2[1-\cos 2(v+g)],$$
$$r^2-3z^2 = [1-6\gamma^2+6\gamma^4]r^2+6\gamma^2(1-\gamma^2)r^2\cos 2(v+g),$$
$$x^2-y^2 = (1-\gamma^2)^2r^2\cos 2(v+g+h)+\gamma^4r^2\cos 2(v+g-h)+2\gamma^2(1-\gamma^2)r^2\cos 2h,$$

$$2xy = (1-\gamma^2)^2 r^2 \sin 2(v+g+h) - \gamma^4 r^2 \sin 2(v+g-h) + 2\gamma^2(1-\gamma^2)r^2 \sin 2h,$$
$$xz = \gamma(1-\gamma^2)^{\frac{3}{2}} r^2 \sin(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2 \sin(2v+2g-h)$$
$$- \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2 \sin h,$$
$$yz = -\gamma(1-\gamma^2)^{\frac{3}{2}} r^2 \cos(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2 \cos(2v+2g-h)$$
$$+ \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2 \cos h.$$

It is then plain that the development of these five factors depends on that of the quantities r^2, $r^2 \cos 2v$ and $r^2 \sin 2v$. Denoting the eccentric anomaly by u, we have

$$\frac{r^2}{a^2} = (1-e\cos u)^2,$$
$$\frac{r^2}{a^2}\cos 2v = \frac{3}{2}e^2 - 2e\cos u + \left(1-\frac{1}{2}e^2\right)\cos 2u,$$
$$\frac{r^2}{a^2}\sin 2v = \sqrt{1-e^2}\,(\sin 2u - 2e\sin u).$$

The constant terms of these functions, in their development in periodic series involving multiples of the mean anomaly, are the same as the constant terms of the right members of the last equations after they have been multiplied by $1-e\cos u$. That is, these terms are severally $1+\frac{3}{2}e^2$, $\frac{5}{2}e^2$ and 0. To obtain the remaining coefficients, we put $s = \varepsilon^{u\sqrt{-1}}$, and $z = \varepsilon^{l\sqrt{-1}}$, and recall the theorem that the coefficient of z^i, in the development of any function S according to powers of z, is the same as that of s^i in the development of

$$\frac{s}{i}\,\frac{dS}{ds}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)},$$

according to powers of s. Moreover, adopting Hansen's notation for the Besselian function, we put $\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \Sigma_i . J_\lambda^{(i)} s^i$,
so that, for positive values of i, we have

$$J_\lambda^{(i)} = \frac{\lambda^i}{1.2\ldots i}\left[1 - \frac{\lambda^2}{1.(i+1)} + \frac{\lambda^4}{1.2(i+1)(i+2)} - \ldots\right],$$

and, for negative values, $J_\lambda^{(-i)} = J_{-\lambda}^{(i)}$.

These functions satisfy the following equation,

$$iJ_\lambda^{(i)} = \lambda\left(J_\lambda^{(i-1)} + J_\lambda^{(i+1)}\right).$$

Whence

$$J_\lambda^{(i-1)} = \frac{i}{\lambda}J_\lambda^{(i)} - J_\lambda^{(i+1)},$$
$$J_\lambda^{(i+1)} = \frac{i}{\lambda}J_\lambda^{(i)} - J_\lambda^{(i-1)},$$

and, by writing $i-1$ for i in the first of these and $i+1$ for i in the second,

$$J_\lambda^{(i-2)} = \frac{i-1}{\lambda}J_\lambda^{(i-1)} - J_\lambda^{(i)},$$
$$J_\lambda^{(i+2)} = \frac{i+1}{\lambda}J_\lambda^{(i+1)} - J_\lambda^{(i)}.$$

Consequently $\quad J_\lambda^{(i-2)} - J_\lambda^{(i+2)} = \frac{1}{\lambda}\left[(i-1)J_\lambda^{(i-1)} - (i+1)J_\lambda^{(i+1)}\right].$

The coefficient of z^i in the expansion of $\frac{r^2}{a^2}$ being equal to that of s^i in

$$-\frac{e}{i}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\left(s-\frac{1}{s}\right)\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)},$$

is

$$-\frac{e}{i}\left[J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)} - \frac{e}{2}\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right)\right],$$

which, by means of the relations between the J functions just given, reduces to

$$-\frac{2}{i^2}J_{\frac{ie}{2}}^{(i)}.$$

Hence we have

$$\frac{r^2}{a^2} = 1 + \frac{3}{2}e^2 - \sum_{i=1}^{i=\infty}.\frac{4}{i^2}J_{\frac{ie}{2}}^{(i)}\cos il.$$

This result may also be obtained from the equation

$$\frac{d^2\frac{r^2}{a^2}}{dl^2} = 2\frac{a}{r} - 2.$$

In like manner we get

$$\frac{r^2}{a^2}\cos 2v = \frac{5}{2}e^2 + \sum_{i=1}^{i=\infty}.\frac{2}{i}\left[\left(1-\frac{1}{2}e^2\right)\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right) - e\left(J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)}\right)\right]\cos il,$$

$$\frac{r^2}{a^2}\sin 2v = \sqrt{1-e^2}\sum_{i=1}^{i=\infty}.\frac{2}{i}\left[J_{\frac{ie}{2}}^{(i-2)} + J_{\frac{ie}{2}}^{(i+2)} - e\left(J_{\frac{ie}{2}}^{(i-1)} + J_{\frac{ie}{2}}^{(i+1)}\right)\right]\sin il.$$

Consequently, if we put

$$H^{(i)} = \frac{2}{i}\left[\left(\cos^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)J_{\frac{ie}{2}}^{(i-2)} - e\cos^2\frac{\varphi}{2}\cdot J_{\frac{ie}{2}}^{(i-1)}\right.$$
$$\left. + e\sin^2\frac{\varphi}{2}\cdot J_{\frac{ie}{2}}^{(i+1)} - \left(\sin^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)J_{\frac{ie}{2}}^{(i+2)}\right],$$

where $\sin\phi = e$, and we agree that

$$H^{(0)} = \frac{5}{2}e^2,$$

we shall have, α denoting any arbitrary angle,

$$r^2\cos(\alpha+2v) = a^2\sum_{i=-\infty}^{i=+\infty}.H^{(i)}\cos(\alpha+il),$$

$$r^2\sin(\alpha+2v) = a^2\sum_{i=-\infty}^{i=+\infty}.H^{(i)}\sin(\alpha+il).$$

We can now write the expansions of the five factors of the terms of R which depend solely on the moon's coordinates:

$$
\begin{aligned}
\frac{r^2-3z^2}{4a^2} &= -\frac{1}{2}(1-6\gamma^2+6\gamma^4)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\cos il \\
&\quad+\frac{3}{2}\gamma^2(1-\gamma^2)\Sigma.H^{(i)}\cos(2g+il), \\
\frac{3}{4}\frac{x^2-y^2}{a^2} &= \quad\frac{3}{4}(1-\gamma^2)^2\Sigma.H^{(i)}\cos(2h+2g+il) \\
&\quad-3\gamma^2(1-\gamma^2)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\cos(2h+il) \\
&\quad+\frac{3}{4}\gamma^4\Sigma.H^{(i)}\cos(-2h+2g+il), \\
\frac{3}{2}\frac{xy}{a^2} &= \quad\frac{3}{4}(1-\gamma^2)^2\Sigma.H^{(i)}\sin(2h+2g+il) \\
&\quad-3\gamma^2(1-\gamma^2)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\sin(2h+il) \\
&\quad-\frac{3}{4}\gamma^4\Sigma.H^{(i)}\sin(-2h+2g+il), \\
\frac{3}{2}\frac{xz}{a^2} &= \quad\frac{3}{2}\gamma(1-\gamma^2)^{\frac{3}{2}}\Sigma.H^{(i)}\sin(h+2g+il) \\
&\quad+3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\sin(h+il) \\
&\quad+\frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma.H^{(i)}\sin(-h+2g+il), \\
\frac{3}{2}\frac{yz}{a^2} &= -\frac{3}{2}\gamma(1-\gamma^2)^{\frac{3}{2}}\Sigma.H^{(i)}\cos(h+2g+il) \\
&\quad-3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma.\frac{1}{i}J^{(i)}_{\frac{ie}{2}}\cos(h+il) \\
&\quad+\frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma.H^{(i)}\cos(-h+2g+il).
\end{aligned}
$$

The summation must be extended to all integral values positive and negative, zero included, for i. When $i=0$ we must suppose that $\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}$ takes the value $-\frac{1}{2}\left(1+\frac{3}{2}e^2\right)$.

It will be perceived that the three first terms of R furnish inequalities whose arguments do not involve the longitude of the moon's node or involve it in an even multiple. The two remaining terms furnish inequalities having an odd multiple of this longitude in their arguments. And it is evident that these statements remain true even when the solar perturbations of the lunar coordinates are taken into consideration. Hence, in deriving any particular inequality, we never have to consider more than three out of the five terms of R. When we propose to neglect the solar perturbations, it can be seen at a glance what terms of the expressions above ought to be retained. Thus, in the case of

Hansen's inequality of 273 years, the argument involving only l without either h or g, it is plain that the first term of $\frac{r^2-3z^2}{4a^2}$ can alone furnish it; and, consequently, we may put, very simply,

$$R = -m''a^2(1-6\gamma^2+6\gamma^4)J'_{\frac{e}{2}}\left[\frac{1}{\Delta^3}-3\frac{z'^2}{\Delta^5}\right]\cos l.$$

And the whole difficulty is reduced to finding, in the development of

$$\frac{1}{\Delta^3}-3\frac{z'^2}{\Delta^5},$$

the terms $\qquad A^{(c)}\cos(18l''-16l')+A^{(s)}\sin(18l''-16l').$

III.

We pass now to the consideration of the development, in periodic series, of the factors of the terms of R which depend on the coordinates of the earth and planet. Let it be required to discover the coefficient $C_{i,i'}$ of $z^i z'^{i'}$ in the development of any periodic function of the eccentric anomalies u and u' of two planets, in the case where i is quite large. We shall suppose that the function has $\frac{1}{\Delta^{2n}}$ for a factor. It is known that

$$\frac{1}{\Delta^{2n}} = N^{2n}[1-2\mathfrak{a}\cos(u-Q)+\mathfrak{a}^2]^{-n}[1-2\mathfrak{b}\cos(u+Q)+\mathfrak{b}^2]^{-n},$$

where N, $\mathfrak{a}$, $\mathfrak{b}$ and Q are functions of u' or l' only, and $\mathfrak{a}$ and $\mathfrak{b}$ are always less than unity. Substituting the imaginary exponential $s=\varepsilon^{u\sqrt{-1}}$, and, to abbreviate, putting $\qquad k=\mathfrak{a}^{-1}\varepsilon^{Q\sqrt{-1}}, \quad k_1=\mathfrak{b}^{-1}\varepsilon^{-Q\sqrt{-1}},$

this equation becomes

$$\frac{1}{\Delta^{2n}} = N^{2n}\left(1-\frac{s}{k}\right)^{-n}\left(1-\frac{\mathfrak{a}^2k}{s}\right)^{-n}\left(1-\frac{s}{k_1}\right)^{-n}\left(1-\frac{\mathfrak{b}^2k_1}{s}\right)^{-n}.$$

Rendering evident the factor $\left(1-\frac{s}{k}\right)^{-n}$, we can then suppose that the function to be developed is

$$\left(1-\frac{s}{k}\right)^{-n}F(s).$$

The coefficient of z^i in the development of this is equivalent to

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} s^{-i}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\left(1-\frac{s}{k}\right)^{-n}F(s)\,du.$$

Let us put

$$f(s) = \varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]F(s);$$

then

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} s^{-i}\left(1-\frac{s}{k}\right)^{-n}f(s)\,du.$$

Since the absolute term of a series of integral powers of a variable is not

changed by substituting for the latter a constant multiple of it, in the expression for C_i we can write ks for s. Thus

$$C_i = \frac{k^{-i}}{2\pi}\int_0^{2\pi} s^{-i}(1-s)^{-n} f(ks)\, du.$$

The difficulty here that the factor $(1-s)^{-n}$ becomes infinite at the limits of the definite integral, is only apparent. For the multiple of s instead of ks may be ps, in which the modulus of p is less than that of k by a very small quantity. In this case we get a tangible result, which is seen to have, as its limit, when p is made to approach k indefinitely, the value which will be presently given.

We now assume that it is possible to expand $f(ks)$ in an infinite series proceeding according to positive integral powers of u.* Let

$$f(ks) = c_0 + c_1 u + c_2 u^2 + \ldots = \Sigma \,.\, c_j u^j.$$

Then

$$C_i = \frac{k^{-i}}{2\pi}\Sigma \,.\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n} c_j u^j\, du.$$

The definite integral

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n}\, du$$

is a function of n and i: with Cauchy we will denote it by $[n]_i$. Then by taking the derivative of the quantity, under the integral sign, j times with respect to i, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n} u^j\, du = (\sqrt{-1})^j D_i^j \,.\, [n]_i.$$

Whence we have the symbolic expression for C_i,

$$C_i = k^{-i} f(k\varepsilon^{-D_i})\,.\,[n]_i.$$

But we have

$$\varepsilon^{D_i} = 1 + \Delta, \quad \varepsilon^{-D_i} = \frac{1}{1+\Delta},$$

Δ here denoting the characteristic of finite differences with respect to the variable i, and not the distance between the two planets. Let

$$\nabla = \frac{\Delta}{1+\Delta}, \text{ then } \varepsilon^{-D_i} = 1 - \nabla.$$

Making these substitutions, we have

$$C_i = k^{-i} f(k - k\nabla)\,.\,[n]_i.$$

By successive integrations by parts, making the integration always bear on the first factor, we find the value of the definite integral,

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n}\, du = [n]_i = \frac{n(n+1)\ldots(n+i-1)}{1.2\ldots i}.$$

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*This is the assumption which leads to the semi-convergent series representing the value of C_i. Its allowableness is shown by the fact of the relative smallness of the definite integral which ought to be added to complete the truncated series, when i is tolerably large and the number of terms taken into account is not too great. As Cauchy has treated this point at length, in his memoir first mentioned above, I have thought it unnecessary to say more about it here.

When the function $f(k - k\nabla)$ is developed in ascending powers of ∇, the general term of C_i will be proportional to

$$\nabla^j . [n]_i = \frac{\Delta^j}{(1+\Delta)^j} \cdot [n]_i = \Delta^j . [n]_{i-j} = [n-j]_i .$$

And, developing the last expression for C_i, and employing accents, attached to f, to denote differentiation of the form of f, we have

$$C_i = k^{-i} \Big\{ f(k)[n]_i - kf'(k)[n-1]_i + \frac{1}{1.2} k^2 f''(k)[n-2]_i \\ - \frac{1}{1.2.3} k^3 f'''(k)[n-3]_i + \ldots \Big\} .$$

This may also be written

$$C_i = k^{-i}[n]_i \Big\{ f(k) - f'(k).k \frac{n-1}{i+n-1} \\ + \frac{1}{1.2} f''(k).k^2 \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)} \\ - \frac{1}{1.2.3} f'''(k).k^3 \frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)} \\ + \ldots\ldots\ldots\ldots \Big\} .$$

We may employ the Γ function to express $[n]_i$, and then

$$[n]_i = \frac{\Gamma(i+n)}{\Gamma(n)\Gamma(i+1)} .$$

In practice, n will have some one of the following series of values,

$$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \text{etc.};$$

and it is well known that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi},\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi},\ \text{etc.}$$

When i is a tolerably large integer, we may use the semi-convergent series

$$\log\Gamma(i+n) = \frac{1}{2}\log(2\pi) + \left(i+n-\frac{1}{2}\right)\log(i+n-1) \\ + M\Big\{ -(i+n-1) + \frac{B_1}{1.2}\frac{1}{i+n-1} - \frac{B_3}{3.4}\frac{1}{(i+n-1)^3} + \frac{B_5}{5.6}\frac{1}{(i+n-1)^5} - \ldots \Big\},$$

$$\log\Gamma(i+1) = \frac{1}{2}\log(2\pi) + \left(i+\frac{1}{2}\right)\log i \\ + M\Big\{ -i + \frac{B_1}{1.2}\frac{1}{i} - \frac{B_3}{3.4}\frac{1}{i^3} + \frac{B_5}{5.6}\frac{1}{i^5} - \ldots \Big\},$$

where M is the modulus of common logarithms, and B_1, B_3, etc., are the numbers of Bernoulli. Thence is derived

$$\log\frac{\Gamma(i+n)}{\Gamma(i+1)}=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)$$
$$-M\left\{n-1+\frac{B_1}{1.2}\left[\frac{1}{i}-\frac{1}{i+n-1}\right]-\frac{B_3}{3.4}\left[\frac{1}{i^3}-\frac{1}{(i+n-1)^3}\right]\right.$$
$$\left.+\frac{B_5}{5.6}\left[\frac{1}{i^5}-\frac{1}{(i+n-1)^5}\right]-\cdots\right\}$$
$$=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)$$
$$-M(n-1)\left\{1+\frac{1}{12}\,\frac{1}{i(i+n-1)}-\frac{1}{360}\,\frac{i^2+i(i+n-1)+(i+n-1)^2}{i^3(i+n-1)^3}\right.$$
$$+\frac{1}{1260}\,\frac{i^4+i^3(i+n-1)+i^2(i+n-1)^2+i(i+n-1)^3+(i+n-1)^4}{i^5(i+n-1)^5}$$
$$\left.-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}.$$

The first term of the last expression for C_i affords a first approximation to its value, correct, so to speak, to quantities of the order of $\frac{1}{i}$. Then

$$C_i=k^{-i}[n]_i f(k).$$

In like manner, the two terms at the beginning afford an approximation correct to quantities of the order of $\frac{1}{i^2}$. Here we can effect a remarkable reduction; for, on comparing the two terms in question with the two first terms of Taylor's theorem, we see that, to the same degree of approximation, we may write

$$C_i=k^{-i}[n]_i f\left(\frac{i}{i+n-1}\,k\right).$$

No more labor is involved in employing this expression than in the preceding.

IV.

In this condition Cauchy leaves the subject, but we may go a step farther. In the cases which come up in practice $f(k)$ is always such a function that successive differentiation immensely complicates it; so that it is scarcely possible to go beyond $f''(k)$. Hence a great deal of labor is saved, if, instead of attempting to calculate $f'(k)$, $f''(k)$, etc., we substitute the calculation of $f(k)$ for several values of the argument k. It is easy to perceive that, in general, all the derivatives $f'(k)$, $f''(k)$, etc., may be eliminated from the expression for C_i. For, cutting the series off at the term which contains $f^{(2p)}(k)$ as a factor, we may suppose that, to the same degree of approximation,

$$C_i=k^{-i}[n]_i\{x_0 f(k-ky_0)+x_1 f(k-ky_1)+\ldots+x_p f(k-ky_p)\},$$

where $x_0, x_1, \ldots x_p$ and $y_0, y_1, \ldots y_p$ are unknowns to be suitably determined.

By developing this expression for C_i in powers of k and comparing it with the previous expression, we get the following system of simultaneous equations for determining the unknowns $x_0, x_1, \ldots x_p, y_0, y_1, \ldots y_p$:

$$x_0 + x_1 + x_2 + \ldots + x_p = 1,$$
$$x_0 y_0 + x_1 y_1 + x_2 y_2 + \ldots + x_p y_p = \frac{(n-1)}{i+n-1},$$
$$x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 + \ldots + x_p y_p^2 = \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_0 y_0^{2p+1} + x_1 y_1^{2p+1} + x_2 y_2^{2p+1} + \ldots + x_p y_p^{2p+1} = \frac{(n-1)\ldots(n-2p-1)}{(i+n-1)\ldots(i+n-2p-1)}.$$

For the sake of brevity we will denote the right-hand members of these equations as a_0, a_1, $a_2 \ldots a_{2p+1}$. The solution of these equations is very elegant. According to the theorem of Bezout, the degree of the final equation, obtained by elimination, would be $= (2p+2)!$ But, as we shall see, the solution depends on that of an equation of the $(p+1)^{\text{th}}$ degree, whose roots are the values of the several unknowns y_0, $y_1 \ldots y_p$; and there is practically but one solution.

Let us suppose that the values of the y's, in any particular solution, are the roots of the equation

$$y^{p+1} + s_1 y^p + s_2 y^{p-1} + \ldots + s_{p+1} = 0,$$

so that

$$s_1 = -(y_0 + y_1 + y_2 + \ldots + y_p),$$
$$s_2 = y_0 y_1 + y_0 y_2 + y_1 y_2 + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_{p+1} = (-1)^{p+1} y_0 y_1 y_2 \ldots y_p.$$

Hence, y_q denoting any one of the y's, we must have

$$y_q^{p+1} + s_1 y_q^p + s_2 y_q^{p-1} + \ldots + s_{p+1} = 0.$$

Now, in the group of equations to be solved, multiply the equation, whose second member is a_{p+1}, by 1, the one, whose second member is a_p, by s_1, and so on until the first equation is multiplied by s_{p+1}. Then, by adding all the equations thus obtained, the first member of the resulting equation vanishes, and we have

$$a_{p+1} + a_p s_1 + a_{p-1} s_2 + \ldots + a_0 s_{p+1} = 0.$$

By cutting off the first equation and adding to the group the equation whose second member is a_{p+2}, and writing x_0 for $x_0 y_0$, x_1 for $x_1 y_1$, and so on, we obtain a group which differs from the former only in the second members. Hence we have, from this group, the equation

$$a_{p+2} + a_{p+1} s_1 + a_p s_2 + \ldots + a_1 s_{p+1} = 0.$$

And, in a similar manner,

$$a_{p+3} + a_{p+2} s_1 + a_{p+1} s_2 + \ldots + a_2 s_{p+1} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{2p+1} + a_{2p} s_1 + a_{2p-1} s_2 + \ldots + a_p s_{p+1} = 0.$$

These $p+1$ linear equations suffice to determine the values of $s_1, s_2 \ldots s_{p+1}$, the coefficients of the equation of the $(p+1)^{\text{th}}$ degree, which has, as its roots, the values of the unknowns y. These values, being obtained and substituted in the first $(p+1)$ equations of the original group, we have a group of $(p+1)$ linear equations for determining the $(p+1)$ unknowns $x_0, x_1 \ldots x_p$. It is plain that all possible solutions of the group of equations are obtained by permuting between themselves the roots of the equation which gives the values of the y's; and, as thus, to each root, corresponds its special value of the x's, and the order in which the several terms of C_i stand, is of no import, it is clear that, practically at least, but one solution exists.

In practice, p never need exceed 2. For $p=0$, the solution has already been given. For $p=1$, we have

$$\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}+\frac{n-1}{i+n-1}s_1+s_2=0,$$

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)}+\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}s_1+\frac{n-1}{i+n-1}s_2=0.$$

The solution of these gives

$$s_1=-2\frac{n-2}{i+n-3}, \quad s_2=\frac{(n-1)(n-2)}{(i+n-2)(i+n-3)}.$$

Thus the equation which contains the values of the y's is

$$y^2-2\frac{n-2}{i+n-3}y+\frac{(n-1)(n-2)}{(i+n-2)(i+n-3)}=0.$$

Whence the two values of y are

$$y=\frac{n-2\pm\sqrt{\dfrac{(2-n)(i-1)}{i+n-2}}}{i+n-3};$$

and the corresponding values of x are

$$x=\frac{1}{2}\left[1\pm\frac{i-n+1}{i+n-1}\sqrt{\frac{i+n-2}{(2-n)(i-1)}}\right].$$

In many cases these values will be imaginary, which, however, does not hinder their use, as k is imaginary.

For $p=2$, we have

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)}+\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}s_1+\frac{n-1}{i+n-1}s_2+s_3=0,$$

$$\frac{(n-2)(n-3)(n-4)}{(i+n-2)(i+n-3)(i+n-4)}+\frac{(n-2)(n-3)}{(i+n-2)(i+n-3)}s_1+\frac{n-2}{i+n-2}s_2+s_3=0,$$

$$\frac{(n-3)(n-4)(n-5)}{(i+n-3)(i+n-4)(i+n-5)}+\frac{(n-3)(n-4)}{(i+n-3)(i+n-4)}s_1+\frac{n-3}{i+n-3}s_2+s_3=0.$$

The solution of these equations gives

$$s_1 = -3\frac{n-3}{i+n-5}, \quad s_2 = 3\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}, \quad s_3 = -\frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)}.$$

The equation, which has, for its roots, the values of the y's, is

$$y^3 - 3\frac{n-3}{i+n-5}y^2 + 3\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}y - \frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)} = 0.$$

By comparing this with the equation for the case where $p = 1$, we readily see what the equation would be for higher values of p.

As an example, suppose it were required to find the coefficient of z^{18} in the expansion of

$$[1 - 2\mathfrak{a}\cos(u - Q) + \mathfrak{a}^2]^{-\frac{3}{2}}.$$

Here the form of $f(s)$ is

$$f(s) = \left(1 - \frac{\mathfrak{a}^2 k}{s}\right)^{-\frac{3}{2}}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]\varepsilon^{\vartheta e\left(s - \frac{1}{s}\right)}.$$

In the first place let two terms in the final expression for C_i be regarded as sufficient, that is, put $p = 1$. Then $i = 18$, $n = \frac{3}{2}$, and the two values of y are

$$y = \frac{-1 \pm 2\sqrt{\frac{17}{35}}}{33};$$

and the corresponding value of x is

$$x = \frac{1}{2}\left(1 \pm \frac{35}{37}\sqrt{\frac{35}{17}}\right).$$

Thus the expression for C_i is

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18}\{1.17865 f(0.9880647k) - 0.17865 f(1.0725413k)\}.$$

The error of this is of the order of $\frac{1}{i^4}$, while, in case $p = 0$, which gives the formula

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18} f\left(\frac{36}{37}k\right),$$

which Cauchy employed, the error is of the order of $\frac{1}{i^2}$.

In case we make $p = 2$, and thus have three terms in the formula for C_i, the roots of the cubic

$$y^3 + \frac{9}{29}y^2 + \frac{9}{31.29}y + \frac{9}{33.31.29} = 0$$

must be found. They are

$$y_0 = +0.00804343, \quad y_1 = -0.04617994, \quad y_2 = -0.27220828.$$

The linear equations for determining the x's are

$$\begin{aligned} x_0 + \quad x_1 + \quad x_2 &= 1, \\ 0.0804343x_0 - 0.4617994x_1 - 2.722083x_2 &= 0.2702703, \\ 0.0064697x_0 + 0.2132586x_1 + 7.409736x_2 &= -0.0772201. \end{aligned}$$

The solution of which gives

$$x_0 = +1.3426685, \quad x_1 = -0.3408857, \quad x_2 = -0.0017828.$$

Thus, in this case, we should have

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18}\{1.3426685 f(0.9919566k) - 0.3408857 f(1.04617994k) - 0.0017828 f(1.2722083k)\}.$$

The error of this formula is only of the order of $\frac{1}{i^6}$.

In further illustration of this method, let us find the value $\mathfrak{b}_{\frac{3}{2}}^{(18)}$ of the coefficient of cos 18θ in the periodic development of

$$(1 - 2\alpha\cos\theta + \alpha^2)^{-\frac{3}{2}},$$

where $\alpha = 0.723332$ the ratio of the mean distances of Venus and the earth from the sun. Here the form of $f(s)$ is simply

$$f(s) = \left(1 - \frac{\alpha}{s}\right)^{-\frac{3}{2}}$$

Let us take the formula where $p = 1$. We have

$$\mathfrak{b}_{\frac{3}{2}}^{(18)} = 2C_{18} = 2\left[\frac{3}{2}\right]_{18}\alpha^{18}\left\{1.17865\left(1 - \frac{\alpha^2}{0.9880647}\right)^{-\frac{3}{2}} - 0.17865\left(1 - \frac{\alpha^2}{1.0725413}\right)^{-\frac{3}{2}}\right\}.$$

The value of $\left[\frac{3}{2}\right]_{18}$ will be found in the table at the end of this memoir. And, on the substitution of the numerical values, we get $\mathfrak{b}_{\frac{3}{2}}^{(18)} = 0.090880$. Delaunay, in his memoir,* has 0.090876.

In the case where the function to be developed contains the anomalies of two planets, after the value of C_i has been obtained corresponding to j points evenly distributed on the circumference with reference to the variable l' or the variable u', the value of $C_{i,i'}$ results by employing the method of mechanical quadratures: the formula in the first case being

$$C_{i,i'} = \frac{1}{j}\Sigma . C_i z'^{-i'},$$

and, in the second,

$$C_{i,i'} = \frac{1}{j}\Sigma . C_i \frac{r'}{a'} s'^{-i'}.$$

In the annexed table are given the common logarithms of the function $[n]_i$, for n as far as $n = \frac{9}{2}$, and for i, as far as $i = 30$. As they have been computed with the ten-figure logarithms of Vega's *Thesaurus Logarithmorum*, it is to be presumed that they are correct, in nearly every case, to half a unit in the last place.

* *Connaissance des Temps*, 1862.

TABLE OF THE VALUES OF LOG $[n]_i$.

i.	$n=\frac{1}{2}$.	$n=\frac{3}{2}$.	$n=\frac{5}{2}$.	$n=\frac{7}{2}$.	$n=\frac{9}{2}$.
1	9.6989700	0.1760913	0.3979400	0.5440680	0.6532125
2	9.5740313	0.2730013	0.6409781	0.8962506	1.0925452
3	9.4948500	0.3399481	0.8170693	1.1594920	1.4283373
4	9.4368581	0.3911006	0.9553720	1.3703454	1.7013386
5	9.3911006	0.4324933	1.0693154	1.5464366	1.9317875
6	9.3533120	0.4672554	1.1662254	1.6977043	2.1313599
7	9.3211273	0.4972186	1.2505463	1.8303299	2.3074511
8	9.2930986	0.5235475	1.3251799	1.9484292	2.4650590
9	9.2682750	0.5470286	1.3921267	2.0548845	2.6077265
10	9.2459986	0.5682179	1.4528245	2.1517945	2.7380602
11	9.2257953	0.5875231	1.5083418	2.2407356	2.8580356
12	9.2073118	0.6052519	1.5594944	2.3229224	2.9691860
13	9.1902785	0.6216423	1.6069190	2.3993107	3.0727266
14	9.1744842	0.6368822	1.6511227	2.4706666	3.1696366
15	9.1597610	0.6511227	1.6925154	2.5376134	3.2607171
16	9.1459727	0.6644866	1.7314334	2.6006651	3.3466317
17	9.1330077	0.6770758	1.7681562	2.6602508	3.4279367
18	9.1207733	0.6889750	1.8029183	2.7167322	3.5051026
19	9.1091914	0.7002560	1.8359186	2.7704171	3.5785315
20	9.0981960	0.7109799	1.8673271	2.8215696	3.6485694
21	9.0877306	0.7211990	1.8972903	2.8704181	3.7155162
22	9.0777464	0.7309589	1.9259355	2.9171615	3.7796337
23	9.0682010	0.7402989	1.9533737	2.9619739	3.8411517
24	9.0590577	0.7492537	1.9797027	3.0050085	3.9002732
25	9.0502837	0.7578539	2.0050085	3.0464012	3.9571780
26	9.0418506	0.7661264	2.0293679	3.0862727	4.0120267
27	9.0337327	0.7740954	2.0528490	3.1247310	4.0649628
28	9.0259073	0.7817822	2.0755129	3.1618728	4.1161153
29	9.0183542	0.7892062	2.0974148	3.1977853	4.1656007
30	9.0110560	0.7963858	2.1186051	3.2325484	4.2135252